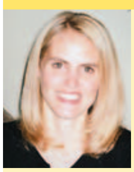
Learning Strategies for Addition and Subtraction Facts:

The Road to Fluency and the License to Think

Teaching the basic facts seemed like the logical thing to do. Wouldn't a study of the basic facts make mathematics computation much easier for my students in the future? How could I help my students memorize and internalize this seemingly rote information? How could I get rid of finger counting and move on to mental computation? As I embarked on my first year of teaching second grade following many years of teaching first grade, these questions rolled through my head.

I had spent the summer poring through the curriculum for grade 2. After familiarizing myself with the mathematics concepts that I would be teaching, I decided to begin the school year with an intense study of strategies for learning and remembering the basic addition and subtraction facts. I looked to *Principles and Standards for School Mathematics* (NCTM 2000) for guidance. I was even more excited about my choice of topics after reading the following quote: "As children in prekindergarten through grade 2 develop an understanding of whole numbers and the operations of addition and subtraction, instructional attention should focus on strategies for computing with whole numbers so that students develop flexibility and computational fluency. Students will generate a range of interesting and useful strategies for solving computational problems, which should be shared and discussed" (NCTM 2000, p. 35). I believed that I was definitely on the right track by focusing on strategies. My next step was to research effective strategies for learning basic facts. *Facts That Last: A Balanced Approach to Mathematics* by Larry Leutzinger (1999a, 1999b) was an invaluable resource. After reading these wonderful books, I set a goal for my students to memorize the basic facts and "move on," but I was in for quite a surprise. I was completely unaware of the impact that this experience would have on my students and their number sense.

By Lisa Buchholz



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The Journey Begins

The first week of school, I began the unit with great excitement. I chose the "doubles" for a starting point. We worked with this concept for a few days; I wanted to be sure that everyone understood it. The first day, we used manipulatives to create equations showing doubles. The next day, we illustrated and wrote about everyday situations in which we can see doubles, such as "5 fingers plus 5 fingers equals 10 fingers" or "2 arms plus 2 legs equals 4 limbs." On the third day, the children wrote their own definitions for "doubles" in their mathematics journals. They

also created equations that showed doubles. My students of average to lower-average ability had equations such as 3 + 3 = 6 and 7 + 7 = 14. My higher-ability students had equations such as 175 + 175 = 350 and 324 + 324 = 648. These students particularly enjoyed the open-ended structure of the mathematics journal assignment, and their enthusiasm ran high. We had begun to really learn and apply doubles instead of merely recognizing them. If I had been paying closer attention, I would have seen the beginning of our number-sense explosion.

After making sure that the students had a full grasp of the doubles, we started working with combinations related to "Doubles Plus One." Using manipulatives, illustrations, and journal writing, we began to train our brains to see "5 + 6" but think "5 + 5 + 1." (I say "we" because I was learning as well. I was taught not to think about equations but to memorize their answers. This "journey" was an awakening for me too.) Soon my students began to compute Doubles Plus One equations mentally, with very little effort. Even my lowest-ability students were solving the equations with ease.

In an attempt to integrate home and school and to share our mathematics excitement and learning, I devised homework for the Doubles Plus One strategy. The assignment asked the children to write an explanation of the strategy and to create problems that could be categorized as Doubles Plus One. The homework also included a parent information sheet, which explained the strategy and gave specific examples of its use. The children were excited to share this homework and show off their new "math brains," a term they had invented.

During our Doubles Plus One exploration, I recalled seeing the same strategy in the district-adopted textbook, which I was not using. The book devoted only one page to the concept of Doubles Plus One. Devoting only one page to this strategy is like expecting someone to successfully ride a two-wheeler after one experience. The sharpest, most capable child might attempt it once and understand all its nuances and complexities, but the majority of children need time and experience to learn and apply the strategy. The page in the book would have presented the strategy, but would the students internalize and apply it or would they only recognize it? In *Elementary and Middle School*

Figure 1

Addition strategies that the students used

Addition Strategies	Our Interpretations and Descriptions
Doubles	Adding two of the same number together, such as 5 + 5 or 7 + 7
Doubles Plus One	Finding "hidden" doubles in expressions where one addend is one more than the other, such as 5 + 6 (thinking 5 + 5 + 1)
Doubles Plus Two	Finding "hidden" doubles in expressions where one addend is two more than the other, such as 5 + 7 (thinking 5 + 5 + 2)
Doubles Minus One	Locating doubles in expressions where one addend is one more than the other, such as 5 + 6 (but thinking 6 + 6 – 1 versus 5 + 5 + 1)
Doubles Minus Two	Locating doubles in expressions where one addend is two more than the other, such as 5 + 7 (but thinking 7 + 7 – 2)
Combinations of Ten	We learned to recognize expressions equaling 10 such as 6 + 4 and 7 + 3 for use in other strategies; we would picture our ten fingers.
Counting Up	This strategy was used only when adding 1 or 2 to a given number; we would see 9 + 2 and think, "9 . . . 10, 11."
Add One to Nine	Used when adding 9 to any number. This was how we "primed" ourselves for the Make Ten strategy; we would see 6 + 9 and think 6 + 10 – 1.
Make Ten	Turning more difficult expressions into expressions equaling 10 and then adding the "leftovers"; we would see 7 + 4 and think 7 + 3 + 1.
Adding Ten	Adding 10 to any number increases the digit in the tens place by one: 5 + 10 = 15, 12 + 10 = 22.
Commutative Property	Any given addends have the same sum regardless of their order: 8 + 7 = 7 + 8.

Mathematics: Teaching Developmentally, Van de Walle (2001) advises not to expect students to have an understanding of an introduced strategy after just one activity or experience. He states that students should use a strategy for several days so they can internalize it. I wholeheartedly agree.

Falling into the Routine

After we studied Doubles Plus One, my lesson plans for the strategies seemed to fall into a consistent pattern. My "mini unit" for each strategy followed this sequence of events:

- Introduce and explore the strategy using manipulatives.
- Create illustrations of the strategy.
- Use mathematics journals: Write a definition of the strategy and create problems that match its criterion.
- Assign homework including a Parent Information Page.

Additionally, each day I read a story problem aloud and asked students to come up with an answer mentally. I called on one student to answer the problem and explain how he or she solved it. Then I asked if anyone had a different way of solving the problem. I called on volunteers until we had heard five or six different ways to solve the problem. If the children had trouble understanding someone's "strategy," I wrote the student's explanation on the board step by step. A chorus of "Oh, I get it now" or "That makes sense" usually followed this illustration. We named this time of day "Mental Math." This was a daily chance for the children to apply their strategies. Over time, I featured two-digit numbers in the story problems. The children simply carried over their knowledge of the strategies to the new problem.

Many of the strategies we studied are featured in Leutzinger (1999a, 1999b) and Van de Walle (2001). We tailored some of these strategies to meet our needs and generated some strategies ourselves. With the combination of the "strategy" lessons (and related activities) and Mental Math sessions, the children demonstrated an amazing command of the world of numbers. They actively used the strategies I had taught them (see **figs. 1** and **2**) and explained them with confidence and conviction, both orally and in written form (see **figs. 3–5**). The children even began to make up their own strategies and explain them with enthusiasm and pride. They named their strategies “Jenna’s Strategy” or “Jack’s Favorite Strategy.” Every day, the knowledge base that we were building became stronger. It was as if I had given my students a license to think. To my students, equations were not just equations anymore; they were numbers that they could manipulate in any way that made sense to them. The following dialogue occurred during a Mental Math session. I have noted the strategies that the students used.

Me. Sarah had 9 fish. Her mother gave her 8 more fish. How many fish did Sarah now have?

Evan. Seventeen, because 9 + 9 is 18. One less would be 17. *[Doubles Minus One]*

Jenna. I know 9 + 9 is 18 and 8 + 8 is 16. . . . In the middle is 17. *[This strategy later became known as Jenna’s Favorite Strategy or The Doubles Sandwich.]*

Jack. I took 2 from the 9 and gave it to the 8, which made 10. Then I took the 7 left over and put it with the 10, which gave me 17. *[Make Ten]*

Peter. Okay, I took 100 plus 100, which gave me 200. I took 200 – 400, which gave me negative 200. Then I took negative 200 + 209, which gave me 9. Then I added 1, which made 10. That’s the Make Ten strategy. Then I added the 7 that were left, which gave me 17. *[Peter loved to work his way away from an answer and come back to it. We named his method the Walk All around the World Just to Cross the Street strategy.]*

Julian. I just knew the answer. *[Fluency]*

Hannah. I started on 9 and counted 8 more. *[Counting Up]*

Mark. I took 8 + 8 and got 16. Then I added 1 more to make 17. *[Doubles Plus One]*

The more strategies we learned, the longer our Mental Math time took. Every minute was worth it. My students seemed to be picturing one another’s strategies mentally. This combination of an intense study of strategies and a daily opportunity for practice added up to success. Even now, well after our study of the strategies has ended (although we still review and practice each day during Mental Math), my students use their solid base of number sense to embrace every new mathematics challenge that comes along.

Another Discovery

Many positive things came out of our journey, such as the number sense and mental mathematics fluency I have already mentioned. I made yet another discovery, however: insight into my students and their abilities. I learned that I could not judge a student by others’ impressions of them. Some students who seemed “lower ability” to their first-grade teachers actually were my best thinkers. They were able to help others understand concepts. Conversely, some students who were named top mathematics students in the previous year had difficulty. These few students were very good at performing algorithms or

Figure 2

Subtraction strategies that the students used

Subtraction Strategies	Our Interpretations and Descriptions
Counting Back	Beginning with the minuend, count back the number you are subtracting; we would see 9 – 3 and think, “9 . . . 8, 7, 6” for an answer of 6.
Counting Up	Beginning with the number you are subtracting, count up to the other number; we would see 12 – 9 and think, “9 . . . 10, 11, 12.” Our answer would be 3 because we counted three numbers.
Doubles	We would see 14 – 7 and think 7 + 7 = 14.
Think Addition	We learned to think of related addition problems when confronted with subtraction facts; we would see 7 – 5 and think 5 + 2 = 7.
Fact Families	Similar to Think Addition above, we would think of the fact family to recall the “missing number.” For a problem such as 8 – 5, we would recall 5 + 3 = 8, 3 + 5 = 8, 8 – 5 = 3, 8 – 3 = 5.
Subtracting from Ten	In equations with 10 as a minuend, we would mentally picture 10 (10 fingers, 10 frames, and so on) to learn what remained when some were taken away.

Figure 3

In her mathematics journal, Tessa demonstrates how a Doubles equation is found in a Doubles Plus One equation.

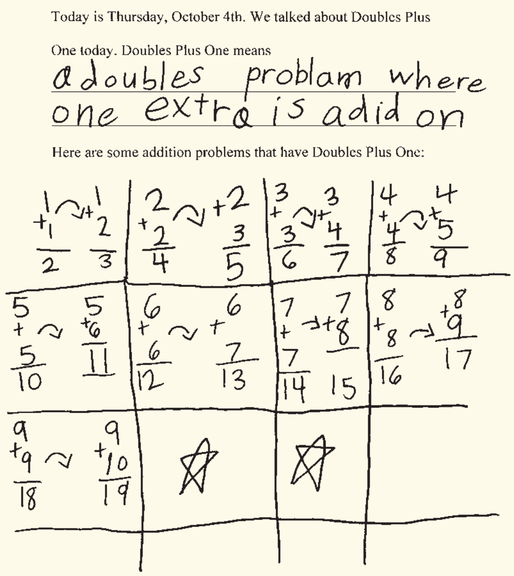
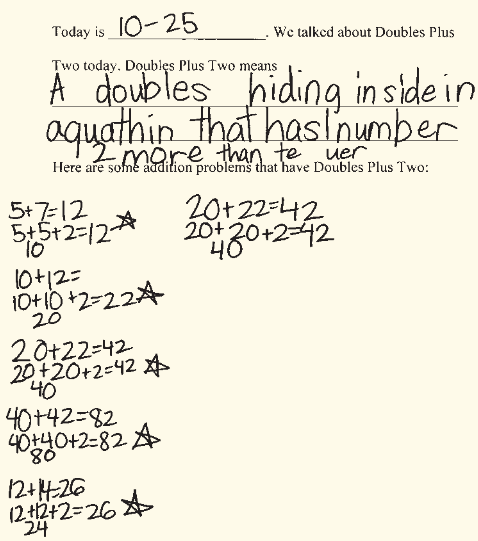


Figure 4

Mary Caroline shows that a Doubles equation is found in a Doubles Plus Two equation.



were already proficient with the addition and subtraction facts. They could not, however, answer the "how" and "why" questions that I posed daily as part of their mathematics journaling, such as "How did you get your answer?" and "Why did this work?" Simply learning what to do is a much easier task than is learning why to do it (Burns 1992).

As the year progressed and we explored new concepts such as renaming, these students' shaky foundation became more apparent. When solving problems involving regrouping, they could not explain why they crossed out a number and renamed it. They did not really understand mathematics; they understood algorithms that they had been taught and had quick recall of the basic facts, showing that they were good at memorizing information. I had uncovered a gap in the mathematics education of these students, a gap that desperately needed to be filled with a better understanding of numbers. A teacher who uses only the mathematics textbook might have a different idea of his or her top mathematics students than would a teacher who probes deeper and expects students to think about their strategies.

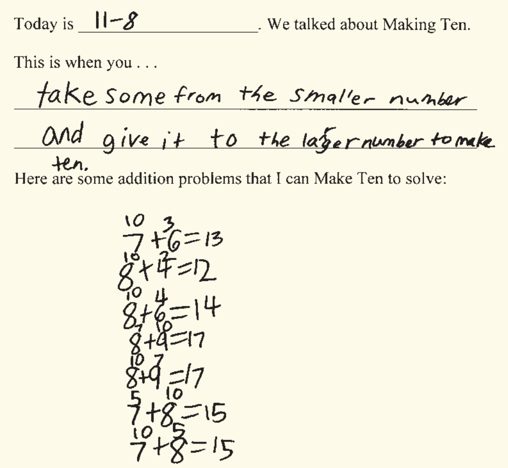
One student, who I will refer to as Steve, fell particularly hard into this mathematics textbook "void." On our "Meet the Teacher" day before school began, Steve's mother informed me that Steve had done third-grade mathematics the previous year. She also told me that he could "borrow" and "carry" to five or six digits. Two weeks into school, she saw Steve struggling and learned that my "lower ability" mathematics students were helping him with his assignments. Steve kept asking, "Can we just do borrowing and carrying? When will we get to borrowing and carrying?" His mother admitted that she wondered what kind of "weird math" I was teaching that had her son so confused. She began to blame his confusion on me and my methods until she saw the hole in her son's mathematics education filled with concepts and strategies. At the end of the year, she stopped in my classroom and said, "Steve has come a long way in understanding numbers. I think he's ready to handle third-grade math now." I firmly believe that if Steve had been taught with a mathematics textbook and a "dabble at the surface" approach, he really would have difficulty later when his ability to perform algorithms was no longer enough.

Final Thoughts

This experience was every teacher's dream. Not only did I grow in my own understanding of numbers but I now have a fresh enthusiasm for teaching mathe-

Figure 5

Sarah shows how the Make Ten strategy works. She begins by creating a problem with two addends equaling more than ten when combined. Then she subtracts from one number to make the other number equal ten.



Today is 11-8. We talked about Making Ten.

This is when you . . .

take some from the smaller number and give it to the larger number to make ten.

Here are some addition problems that I can Make Ten to solve:

7+6=13
8+4=12
8+6=14
8+9=17
8+9=17
7+8=15
7+8=15

matics. I have transformed from a "page a day" mathematics teacher to a facilitator of mathematics and its concepts.

This adventure into number sense took us about two months. In today's crowded curriculum, that is a considerable amount of time to invest in just one concept. My students emerged from this study, however, as amazing thinkers ready to take on any challenge that comes their way. The experience helped my students become faster and more accurate with mathematics flash cards and timed assessments, but the real gain was in number sense.

References

Burns, Marilyn. *About Teaching Mathematics.* Sausalito, Calif.: Math Solutions Publications, 1992.

Leutzinger, Larry. *Facts That Last (Addition): A Balanced Approach to Memorization.* Chicago: Creative Publications, 1999a.

———. *Facts That Last (Subtraction): A Balanced Approach to Memorization.* Chicago: Creative Publications, 1999b.

National Council of Teachers of Mathematics (NCTM). *Principles and Standards for School Mathematics.* Reston, Va.: NCTM, 2000.

Van de Walle, John A. *Elementary and Middle School Mathematics: Teaching Developmentally.* New York: Addison Wesley Longman, 2001.