Learning Continuum for Multiplication and Division

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- CBA materials are designed to help students move to higher levels of reasoning. It is important, however that instruction not *demand* that students 'move up" the levels of sophistication with insufficient cognitive support.
- "Jump in levels" are made internally by students, not by teachers or the curriculum. This does not mean that students must progress with no help. Teaching helps students by providing them with the right kinds of encouragement, support and challenges.
- Good teaching has students work on problems that stretch, but do not overwhelm, their reasoning; asks good questions, has students discuss their ideas with other students; and sometimes shows students ideas that they don't invent themselves.
- Two major components in the development of students' understanding and proficiency with multiplication and division:
    - First students must understand what the operations mean and recognize when each is appropriate in problem solving.
    - Second, students must understand and become proficient with strategies for performing computations for these operations.
- Multiplication situation: We are given the number of equivalent groups and the number in each group and asked to find the total.
    - (number of equivalent groups) X (number in each group) = total
- Two major division situations:
    1. Measurement Division – we are given the total number of objects and asked how many equivalent groups of a given size can be made from the total.

(number of objects) ÷ (number of objects in a group) = (number of equivalent groups)

2. Partitive Division – we are given the total number of objects and asked how many objects will be in each group if the objects are partitioned (divided, separated) into a given number of equivalent groups.

(number of objects) ÷ (number of equivalent groups) = (number of objects in a group)

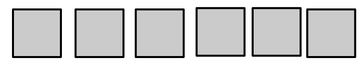
- Many students are puzzled by the two different meanings for division. It is often helpful to these students to see how the measurement and partitive meanings are related for a specific problem.
- Students begin to develop an understanding of multiplication by iterating (repeating or accumulating) numerical composite units. (a composite unit is a collection of things that has been mentally combined and treated as a unit)
- To use iterative reasoning for multiplication, students must be able to coordinate simultaneous counting sequences – one for the number of objects in

a group, one for the number of groups and one for the total number of objects in accumulating iterating of groups.

- Coordinating these counting sequences is difficult, and lack of coordination is a major source of student errors.
- To understand the various meanings of multiplication and division, students must also understand the critically important inverse relationship between multiplication and division.
- A very useful consequence of the inverse relationship between multiplication and division is what we can solve division problems by solving related multiplication problems.
- Students must develop a good deal of proficiency with single-digit (SD) multiplication and division before progressing to multidigit (MD) multiplication and division.
- Before students progress to multidigit multiplication and division, they should develop sufficient fluency with the "basic facts" for multiplication and division. Without adequate fluency with these facts, the cognitive demands require to implement multidigit multiplication and division will be too great for most students to handle.
- Nevertheless, students do not need complete mastery of basic facts before they can make sense of multidigit strategies.
- Although reasoning about multiplication generally develops before reasoning about division, students can often learn both operations at the same time.
- An important part if this development is understanding and becoming fluent with using computational algorithms. *However, if algorithms are taught too early in students' development of reasoning about addition and subtraction, students cannot understand the algorithms conceptually, so they learn by rote.*
- Implementing reasoning strictly verbally is more sophisticated than implementing it concretely or pictorially.
- When using place value the prominent feature should be their numeric value not their shape. Thus when we are verbally describing them use the terms "ten" or "ten block" not "strip" or "rod". The goal in using place-value blocks is to help students develop reasoning about numbers, not blocks.

Levels of Sophistication in Student Reasoning: Multiplication and Division

- These levels describe how students progress from beginning understanding of multiplication and division concepts to meaningful use of multiplication and division algorithms.
- The "jumps" between sublevels are small enough that students can achieve them with small amounts of instruction in relatively short periods of time. Sublevels serve as accessible stepping-stones in students' development.

SD Level 0	Student does not understand multiplication and division situations. (students do not understand that groups of objects, not just single objects, can be counted)	
	Task: Jon has 5 cans of tennis balls. Each can has 3 balls in it. How many tennis balls does Jon have altogether? Response: I'd say 8.	Task: How many groups of 2 squares are there?  Response: 1,2,3,4,5,6. (The student counts individual objects instead of groups of objects)
SD Level 1	Student multiplies or divides by counting objects in groups by ones with no skip-counting. (Although students understand multiplication and division they solve problems by counting by one)	
1.1	Student counts physical or visual objects by ones. (Students form equal groups of things they can touch or see)	
	Task: 6 × 8 Response: I made 6 piles of 8 cubes. 1,2,3,... 46, 47, 48 (The student uses cubes to model the problem and then counts all the cubes by ones)	Task: 18 ÷ 3 Response: Student counts out 18 blocks and puts them into 3 equal groups to get 6.
1.2	Student correctly counts visualized objects or counting words by ones. (Student correctly solve problems using counting words but may use physical objects to represent the number of groups but not objects themselves.)	
	Task: 6 × 8 Response: The student taps on the desk 8 times (1,2,3…8) and then holds up one finger. they continue the process until they reach 48.	Task: I have 20 cubes. I want to put them into containers so there are 5 cubes in each container. How many containers do I need? Response: 1,2,3,4,5,(raises 1 finger) 6,7,8,9,10 (raises 2nd finger) …16, 17, 18, 19, 20 (raises a 4th finger). 4 containers.
1.3	Student uses uncoordinated, incorrect skip-counting. (Student no longer counts by one but has difficulty keeping track of the two number schemes)	
	Task: I have 4 containers. there are 3 cubes in each container. How many cubes are there altogether? Response: 1 group is 3 (puts up one finger), 2 groups is 6 (puts up a second finger), 7 (puts up a third finger), 8 (puts up a fourth finger).	Task: I have 12 cubes. I want to put them into containers, so there are 3 cubes in each container. How many containers do I need? Response: Student says 3 (holding up 3 fingers), 4 (holding up another 3 fingers), 5 (holding up 3 fingers) and 6 (holding up 3 fingers once again).

SD Level 2	Student multiplies/divides numbers by repeated addition/subtraction or skip-counting (repeated addition and subtraction and skip counting are two efficient methods for multiplying and dividing single digit numbers)	
2.1	Student uses repeated addtion or subtraction, or skip-counting and counts by ones. (student either repeatedly add or subtract or they skip count some numbers then count by ones)	
	Task: I have 4 containers. There are three cubes in each container. How many cubes are there altogether? Response: (extends 1 finger) 3, (extends 2nd finger) 6, (extends 3rd finger) 9, (extends 4th finger) 10, 11, 12 (The student skip counts part of sequence but then completes the sequence by counting by ones)	Task: 18 ÷ 3 Response: 18 – 3 = 15 15 – 3 = 12 12 – 3 = 9 9 – 3 = 6 6 –3 = 3 3 – 3 = 0 So 6.
2.2	Student decomposes a number into parts and skip-counts those parts (Not a goal for instruction because it is complex for students to manage without making errors)	
	Task: 6 × 8 Response: 5,10,15,20,25,30 plus 3,6,9,12,15,18. 30 plus 18 is 48 (decompose 8 into 3 and 5)	Task: 84 ÷ 4 Response: 4 plus 4 is 8, 8 plus 8 is 16, So that is 4 fours. 16 and 16 is 32, that's 8 fours. 64 makes 16 fours, plus 16 is 80 which is 20 fours. One more four makes 21.
2.3	Student skip-counts all multiples in the skip-count sequence without decomposing numbers into parts. (Students multiply or divide by counting all of the numbers in the skip-count sequence)	
	Task: I have 4 containers. There are three cubes in each container. How many cubes are there altogether? Response: (Writes 3 and 1 underneath it, 6 and 2 underneath, etc…) There are 12 cubes.	Task: 38 ÷ 5 Response: Raising fingers one at a time 5,10,15,20,25,30,35. Seven 5's and there's a remainder of 3.
2.4	Student skip-counts a group of skip-counts (Although this procedure is quite clever it is often difficult for students to keep track of what they are doing so it is not encouraged)	
	Task: 9 × 4 Response: 4,8,12 that's three groups of 4. Three more 4's would be 24 and three more would be 36.	Task: 40 ÷ 5 Response: You count by 5s. 5, 10, that's two 5s in 10. There are 4 tens in 40 so you would have 8.

Students may use Level 2.2 and 2.4 reasoning as they gain skip-count fluency, these levels are not a prerequisite for making progress toward Level 3.

<table>
<tr><td>SD Level 3</td><td colspan="2">Student multiplies or divides numbers by recalling facts or by using properties to derive answers from known facts with no counting or skip-counting. (A fact is meaningful rather than rote if the student can justify the fact y counting, using physical or pictorial models or deriving the fact from other facts. once you have established that a student knows his or her facts meaningfully, you do not need to continue to as such questions.</td></tr>
<tr><td>3.1</td><td colspan="2">Student directly recalls basic multiplication or division facts.</td></tr>
<tr><td></td><td>Task: I have 4 containers. There are three cubes in each container. How many cubes are there altogether?

Response: 4 times 3 is 12. [Teacher: How did you get 12?] If you take 4 + 4 + 4 you get 12. [Did you do it this way?] No I just know that 4 times 3 is 12.</td><td>Task: $38 \div 5$

Response: Well 7 times 5 equals 35 and there’s a remainder of 3.</td></tr>
<tr><td>3.2</td><td colspan="2">Task: Student uses number properties to derive answers from known facts. (Student generally will not know the name of the number property but will have learned the property intuitively)</td></tr>
<tr><td></td><td>Task: 6×8
Response: 5 times 8 is 40 plus one more 8 is 48.

Number properties:
▪ Commutative Property:
$6 \times 8 = 8 \times 6$

▪ Associative Property:
$5\times6 = 5 \times (2\times3) = (5\times2) \times 3 = 10\times3 = 30$

▪ Distributive Property:
$7 \times 8 = 7 \times (5+3) = (7 \times 5) + (7 \times 3)$
$= 35 + 21 + 56$

$45 \div 5 = (25 + 20) \div 5 = (25\div5) + (20\div5)$
$= 5 + 4 = 9$</td><td>Task: $40 \div 5$
Response: Since 5×8 is 40, $40 \div 5$ is 8.

▪ Inverse Relationship:
$48 \div 6 = 8$ since $6 \times 8 = 48$

▪ Division/ Multiplication Property:
$42 \div 6 = 42 \div (2\times3) = (42 \div 2) \div 3$
$= 21 \div 3 = 7$

$7 \times 5 = 7 \times (10 \div 2) = (7 \times 10) \div 2$
$= 70 \div 2 = 35$</td></tr>
</table>

- When you are sure that students understand the meaning of multiplication and division and they have started to remember many of the facts, it is important for students’ mathematical fluency to help them practice these facts.

Develop a Profile of a Student's Reasoning about Addition and Subtraction

- CBA assessment tasks are designed to help you assess levels of reasoning, not levels of students.
- A student might use one level of reasoning of visually presented tasks, and another on symbolic tasks.
- CBA Reasoning Profile
    1. Record what the student did on CBA problems
    2. Construct a CBA levels summary chart for student.
    3. Develop a set of goals and recommendations for instruction.

CBA Assessment Tasks for Addition and Subtraction

- The problem sheets can be used in individual interviews with children or in class as instructional activities. However, no matter what you choose, it is critical to get the students to write and describe or discuss their strategies. Only then can you use the CBA levels to interpret students' responses and decide on needed instruction.
- The purpose of interviewing students with CBA tasks is to determine how they are reasoning and more specifically, to determine what CBA levels of reasoning they are using for the tasks.

CBA Multiplication and Division

Student Sheet 1

1. George has 4 cups. There are 2 cubes in each cup. How many cubes does George have altogether?

2. Greg has 5 cups. There are 3 cubes in each cup. How many cubes does Greg have altogether?

3. Marge has 5 bags of candy. There are 4 pieces of candy in each bag. How many pieces of candy does Marge have altogether? Use cubes to act out the problem.

4. Mary has 6 cubes. She puts 2 cubes in each cup. How many cups does she have?

5. Zach has 12 cubes and 3 cups. He puts the same number of cubes into each cup. How many cubes are in each cup?

6. Mimi has 15 pennies. She wants to put them into 5 equal stacks. How many pennies will be in each stack? Use cubes to act out the problem.

7. Mimi has 18 pieces of candy. She wants to put 3 pieces in each bag. How many bags does she need? Use cubes to act out the problem.

Instructional Strategies for Multiplication and Division

- For students to make progress, have them do several problems of a specific type until you see them move to the next level or you become convinced that they are not quite ready to move on to the next level.
- Attaining higher levels of reasoning in multiplication and division, especially the highest levels, is interrelated with attaining higher levels of reasoning about place value and the properties of numbers.
- Student's levels of reasoning for division will generally lag behind their level for multiplication.
- Students should develop a great deal of proficiency with single-digit multiplication and division before progressing to multidigit multiplication and division.

Teaching Students at SD Level 0: Counting Initial Meaning for Multiplication and Division

- Give students problems that involve multiplication and division in physical situations. Placing equal sets of objects in physical containers can help understand that groups, not just individual objects, are countable.
- At first, you may have to ask guiding questions to help students understand the physical situation.
- Once students understand situations about cubes and cups, give problems that are not cubes and cups but can be acted out with cubes.
- You may have to provide similar guidance for division problems.

Teaching Students at SD Level 1: Increasing the Sophistication of Counting by Ones

Level 1.1: Moving to Counting Imagined Objects or Counted Words

- To help students progress beyond using physical objects, it is quite helpful to have them explicitly try to imagine or visualize physical situations.
- if they cant solve the problem by imagining I, suggest that they draw a picture that will help them solve it. If they cannot solve the problem by drawing a picture (or if they get the incorrect answer), give them the cubes and cups.
- Having students gain facility working with imagined groups of objects is an excellent way for them to progress to counting words.
- Ask students if they can solve the problem without using cubes or pictures,. If students cannot solve the problem, suggest they count. If they still can't solve, suggest they use their fingers. If they still can't solve it, you or another student might demonstrate the counting count words strategy.
- Our **next goal** is for students to progress to iteration by skip-counting. Move through various combinations of skip-counting, counting by ones, repeated addition and subtraction, and skip counting by parts.

Levels 1.2 or 1.3: Beginning Skip-Counting

- Ask students if they can solve the problems without counting all the numbers by ones. Students might use repeated addition or subtraction or skip-counting, or a combination on the problems.
- As students begin learning skip-count sequences, they often know the first several multiples in the sequence but have to figure out later ones.
- Ask students who do not spontaneously use addition or counting by ones to continue to an unknown skip-count sequence if counting by does or adding will help them.
- Students may progress to skip-counting in some contexts (with concrete materials) before other contexts (drawing or using count words).

Levels 1.3, 2.1 or 2.2: Progressing to Fluent Skip-Counting (2.3)

- At first, help students link verbal skip-counts with visual iterations of the sets. It is also helpful to use multiples arrays.
- Once students understand skip-counting, give problems in which the objects to be skip-counted are not visible.
- If students can't use skip-counting, use pictures to illustrate the connection between counting the number of packs and skip counting the number of pieces.
- Once students have developed fluency skip-counting to solve multiplication problems, help them extend this procedure to division problems.

Levels 2.3 and 2.4: Moving to Using Number Facts

- Students solving problems using reasoning in Levels 1 and 2, they are meaningfully solving these problems because the problem either:
    - Focus on physical contexts that make sense to students or
    - Build on already understood mathematical procedures (such as counting or adding)
- If students do enough of those problems in Level 1 and 2, they will naturally start remembering some problem answers.
- Introducing multiplication and division notation and language to describe these problems and their answers can help students consolidate and recall these "basic facts".

Teaching Students at SD Level 3: Moving to Understanding and Using Number Properties and Facts.

Level 3.1: Moving to Using Number Properties to Derive Facts

- Once students know some multiplication and division facts, they can begin using number properties to derive answers to related problems. They discover these properties by looking for patterns as they solve carefully chosen problem sequences or by looking carefully at physical or pictorial representations.

- At this age, learning the algebraic representations is not nearly as important as acquiring an intuitive, conceptual understanding of these ideas.
  - Commutative Property: Have students compute answers to pairs of multiplication problems with the order of factors reversed. To help students understand why the commutative property for multiplication is true, you can use arrays.
  - Associative Property: One way of using the associative property is to change a problem into an alternate problem.
  - Inverse Relationship: Help students discover the important relationship between multiplication and division patterns in sets of problems.
  - Distributive Property: Understanding the distributive property is critical to multiplying and dividing, especially when the operations involve multidigit numbers. You can illustrate the distributive property with grouping diagrams.