Learning Continuum for Addition and Subtraction

From *Cognition-Based Assessment and Teaching of Addition and Subtraction* by Michael Battista, 2012.

- CBA materials are designed to help students move to higher levels of reasoning. It is important, however that instruction not *demand* that students 'move up" the levels of sophistication with insufficient cognitive support.
- "Jump in levels" are made internally by students, not by teachers or the curriculum. This does not mean that students must progress with no help. Teaching helps students by providing them with the right kinds of encouragement, support and challenges.
- Good teaching has students work on problems that stretch, but do not overwhelm, their reasoning; asks good questions, has students discuss their ideas with other students; and sometimes shows students ideas that they don't invent themselves.
- Two major components in the development of students' understanding and proficiency with addition and subtraction:
    - First, students must develop an understanding of the meanings of the operation of addition and subtraction.
    - Second, students must develop an understanding of and proficiency with methods for adding and subtracting numbers utilizing a variety of reasoning strategies and culminating in understanding paper-and-pencil computational algorithms.
- To understand adding and subtracting numbers, students must first understand numbers as telling how many objects are in sets, which they determine primarily by counting. For small sets, students often use subitizing (instantly recognize the number of objects in a set).
- Once students understand numbers, they can progress to developing conceptual understanding of physical situations that give rise to addition and subtraction. There are three basic situations.
    1. Addition – Students understand addition in terms of physically joining sets of objects. A somewhat more abstract addition situation occurs when two sets of objects are mentally joined rather than physically joined.
    2. Subtraction – Initially, students understand subtraction in terms of physically taking away one set of objects from another. It has been noted that the phrase "take away" has a negative connotation for many children and instead teachers should talk about "giving".
    3. Comparison – "Johnny has 8 black marbles and Emily has 5 gray marbles. How many more marbles does Johnny have?" These problems can be solved using either addition or subtraction and serve to introduce the idea that addition and subtraction are inverse operations.
- It is extremely important that students develop an understanding of the inverse relationship between addition and subtraction.
- The levels of sophistication in students' reasoning follows this general progression:
    1. Students use physical materials to model the action or relations described in problems and use counting to solve them. Students use count only strategies; first students count by ones and then by place value parts.

2. Students then combine and decompose numbers by place value parts without counting.
3. Students understand and use symbolic algorithms.

- A computational algorithm is a precisely specified sequence of actions performed on written symbols that systematically solves general type of computation problem. *However, if algorithms are taught too early in students' development of reasoning about addition and subtraction, students cannot understand the algorithms conceptually, so they learn by rote.*
- Implementing reasoning strictly verbally is more sophisticated than implementing it concretely or pictorially.
- When using place value the prominent feature should be their numeric value not their shape. Thus when we are verbally describing them use the terms "ten" or "ten block" not "strip" or "rod". The goal in using place-value blocks is to help students develop reasoning about numbers, not blocks.

Levels of Sophistication in Student Reasoning: Addition and Subtraction.

- These levels describe how students progress from a beginning understanding of addition and subtraction concepts to meaningful use of addition and subtraction algorithms.
- The "jumps" between sublevels are small enough that students can achieve them with small amounts of instruction in relatively short periods of time. Sublevels serve as accessible stepping-stones in students' development.
- CBA levels focus on the development of concepts and reasoning in addition to computational fluency. The CBA levels develop gradually as you study examples of student's work and as you use CBA with your students.

Level 0	Students do not understand addition and subtraction situations. They may be able to count single sets, but they do not understanding joining/separating parts.
	Task: Jim has 5 pencils. His teacher gives him 2 more pencils. How many pencils does Jim have? Response: He has 5 pencils Task: Jocelyn has 5 cubes. Brenda has 1 more cube than Jocelyn. How many cubes does Brenda have? Response: One. [Teacher: how did you figure that out?] It says Brenda has one cube.

<table>
<tr><th>Level 1</th><th>Student adds or subtracts numbers as collections of ones)no skip counting by place value) (no use of rote or use of algorithms)</th></tr>
<tr><td>1.1</td><td>Student counts all. (counts by ones for both given numbers)
Students start by counting all visible objects.
Some students may be able to count using imagined objects. To assist in this transition some students may need to see the objects first and they can be hidden.
Counting using fingers is more difficult than counting using objects because students must produce and operate on the representations they have created.
Counting using counting words is more difficult as students must keep track of how many counting words they have used. Students may use their fingers.
Task: 3 + 8
Response: Student makes one set of 3 cubes and one set of 8 cubes. She pushes them altogether then counts them, “1,2,3,4…10,11”.

Task: 15 + 9
Response: 1,2….15. [student puts up 10 fingers and counts 16-24 while folding down 9 fingers] She needed fingers to keep track of her count of the second set of counting words.</td></tr>
<tr><td>1.2</td><td>Student counts on or counts down by ones.
Students give answers that are off by one because they do not completely understand what reciting numbers means in this context.
Requires a major jump in abstraction, as student must recognize that a number exists even when they are not involved in counting it.
This is difficult for students as they must coordinate two number sequences simultaneously. The counting sequence and the number of items to be counted.
Sophistication: visual or physical, counting on fingers, counting words.
Task: 3 + 8
Response 1: (putting up 8 fingers, one at a time) 4,5,6,7,8,9,10,11.
Response 2: (putting up 8 fingers, one at a time) 3,4,5,6,7,8,9,10. – off by one error

Task: Jon has 19 pens. He got 3 more pens for his birthday. How many pens does Jon have now?
Response: 19 plus one would equal 20, so that would be one pen. An then 20 plus another 1 would equal 21; that’s another pen. And then 21 plus 1 would equal the answer which is 22, and that’s 3 pens.</td></tr>
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<tr>
<td>1.3</td>
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Student recalls or derives facts using number properties.

Students decompose or combine known facts to derive unknown facts.
When students use a number property, they usually cannot name or verbally describe the property.
Number properties for deriving unknown facts form know facts are the association, commutative properties and the inverse relationship between addition and subtraction. (fact families)

Task: What is 8 +8
Response 1: [immediately] 16 [Teacher: How did you get that?] Memory
Response 2: 16. Its in our math sheets all the time.
Response 3: 16. I know because it’s a basic fact.

Task: What is 8 plus 9?
Response1: 17. Because 8+8 = 16, and 9 is 1 more than 8, so add 1 and get 17.
Response 2: 17,Because if it was 10, it would be 18, but since its 9, it’s minusing 1, so it’s 17.
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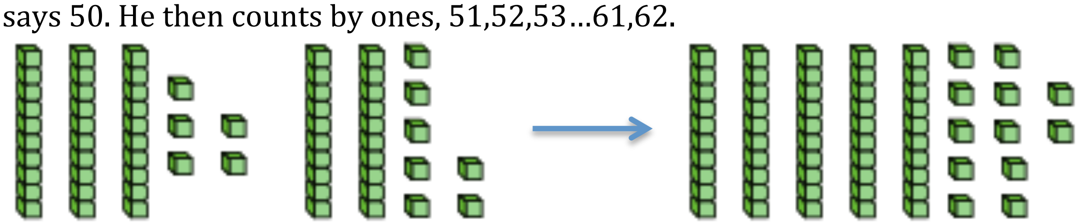
<tr>
<td>1.4</td>
<td>
Student operates on tens and ones separately.

They don’t understand the relationship between tens and ones.
When counting using ten-blocks and one-blocks, students count all the one-blocks (either separate or embedded in ten-blocks)

Task: 47 + 27
Response 1: [Makes 47 with 4 ten-blocks and 7 one-blocks] 47. [Makes 27 with 2 ten-blocks and 7 one-blocks] 27.] Pushes all the base-ten blocks together; counts by ones, first all the ten-blocks, then the one-blocks] 1,2,3…73,74.
Response 2:
$$47 + 27 =$$
[Adding 7 and 7] 14. So, its 4 [Writes 4 on her paper] Put the 1 here [Writes 1 above the 4 on 47] Then [Adding the digits in the tens columns] you get 7, because 4; 5,6,7. [Writes 7] Okay, 74. [Teacher: Why did you put the 1 above the 4?] Because that’s where you are supposed to put it. [Teacher: Is this really a 1?] Yes, you just add 1 to 4 and 2.

Task: In a box, there are 35 red apples and 27 green apples. How many apples are in the box?
Response: Student makes 35 with base-ten blocks, then 27 with base-ten blocks, puts the ten-blocks together and one-blocks together. Student lays his hand on the 5 ten-blocks and says 50. He then counts by ones, 51,52,53…61,62.

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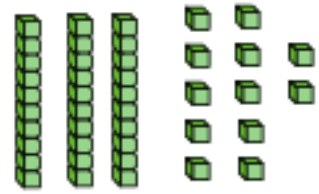
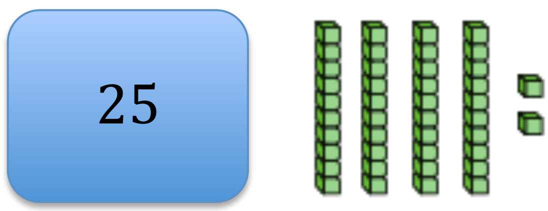
Level 2	Student adds or subtracts numbers by skip-counting place-value parts.
2.1	Student counts by tens and ones separately.  Task: In a box there are 17 green apples and 25 red apples. How many apples are in a box? Response: Student represents each number. She then moves all the ten-blocks and one-blocks together. 10, 20, 30, 31,32…41, 42. Task: In a box there are 35 red apples and 27 green apples. How many apples are in a box? Response 1: [using fingers] 10,20,30,40,50;51,52,53,54,55,56,57,58,59,60,61,62 Response 2: 30;40,50,55,60,62 Response 3: 30;40,50,55,56,57,58,59,60,61,62
2.2	Student counts by tens in mid-decades. Task: There are 25 squares under the card. How many squares are there altogether?  Response: [After counting ten squares on one ten-block] 25;35,45,55,65;66,67. Task: 39 + ____= 71 Response: 39,49,59,69;70,71. o the number that goes here [in the blank] is 42. As this response shows, for more complex problems, especially when students must keep track of more than one counting sequence, it is not uncommon for students to lose track of their counting.

Level 3	Student adds or subtracts by combining or separating place-value parts. (Explicit use of place value in informal multidigit arithmetic; emerging but incomplete understanding of place-value in algorithms)
3.1	Student uses tens landmarks. Task: 25+37 = Response: I did it on a number line [going left to right]
3.2	Student separately adds or subtracts the tens and ones part of one number to or from the other number. • Students might add or subtract multiples of ten close to the given numbers then compensate for the fact that the multiples of ten were not the original numbers. Task: 17 + 25 Response: 20 + 25 is 45, but it is 17, so you have to subtract 3. It's 42. Task: George has 24 dollars. How many more dollars does he need so that he has 47 dollars? Response: Add 20 to 24 to get 44 then add 3 to 44 to get 47. So 30 plus 3 equals 23.
3.3	Student combines or separates the tens parts and ones parts and then adds the results. (Counting is not the primary strategy) • Students may use physical materials but not in a way that is consistent with the traditional algorithm. (May not represent the numbers first or may not regroup). These discrepancies are one reason students may have difficulty connecting their thinking with the traditional algorithm. • Using only a verbal approach, students at this level always add or subtract then tens without counting, but they might add, subtract or count ones. Task: 17 + 25 Response: 7 plus 5 is 12. 10 plus 10 is 20. So, if you add the 10 from the 12 to the 30 you get 40 then add the 2 from the 12 to get 42. Task: 36 + 28 Response 1: [Student gets 3 ten-blocks, then 2 ten-blocks, then 8 one-blocks, then 6 one-blocks. She counts the 8 one-blocks and 2 of the 6 one-blocks:] 8;9,10. [Then she trades these 10 one-blocks for 1 ten block. She pushes all the ten-blocks together] 60. [Then she puts the 4 one-blocks with the ten-blocks and answers] 64. Response 2: [Makes 36 with 3 ten-blocks and 6 one-blocks then 28 with 2 ten-blocks and 8 one-blocks.] All that together is 50. [Putting the ten-blocks together] and put the 14 ones together. SO you add 50 and 14 and you get 64.

<table>
<tr><th>Level 4</th><th>Student uses and understands expanded addition and subtraction algorithms. (place-value understanding of expanded algorithms through hundreds)</th></tr>
<tr><td></td><td>Task: 17 + 25
Response:
<table>
<tr><td>All Place-Value Parts Explicitly shown</td><td></td><td>Place-Value Parts Explicit in Verbal Description</td><td></td></tr>
<tr><td>10 + 7
+ 20 + 5
30 + 12 = 42</td><td>7 plus 5 is 12.
10 plus 20 is 30.
30 plus 12 is 42</td><td>17
+ 25
12
30
42</td><td>7 plus 5 is 12.
10 plus 20 is 30.
30 plus 12 is 42</td></tr>
</table>

Students operating at this level reliably use the expanded algorithm
Generally students learn addition algorithm before subtraction
These are very important since they reduce the cognitive load making it less likely that students will make errors and it provides students the conceptual knowledge they need to understand traditional algorithms.
To understand these algorithms students must understand place-value of each digit and understand how the tens and ones are combined to make the whole two-digit number.
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<tr><th>Level 5</th><th>Student uses and understand traditional addition and subtraction algorithms (Place-value understanding of traditional algorithms through hundreds)</th></tr>
<tr><td></td><td>Task: 17 + 25
Response: 7 plus 5 is 12. [Writes the 2 below the 5, writes the 1 above the 1] 1 plus 1 plus 2 is 4. It is 42.

Place-value is dealt with implicitly by the placement of the digits, not in the language, so place value are hidden.
Students can provide a conceptually based explanation of why steps they perform are valid.
To determine the level of understanding, teachers need to ask probing questions such as "Is this really 1?”
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- Students need to be at 3.3 prior to introducing algorithms. Often algorithms are introduced too early. Students may exhibit four different types of understanding of algorithms.
    1. Student incorrectly performs the algorithm. Often conceptual errors are very apparent
    2. Student correctly performs the algorithm, but with a clear indication that it is not understood. (Responds incorrectly to probing questions)
    3. Student correctly performs the algorithm with no indication of understanding or misunderstanding (Probing questions are needed)
    4. Student correctly and meaningfully uses the algorithm.

Develop a Profile of a Student's Reasoning about Addition and Subtraction

- CBA assessment tasks are designed to help you assess levels of reasoning, not levels of students.
- A student might use one level of reasoning of visually presented tasks, and another on symbolic tasks.
- CBA Reasoning Profile
    1. Record what the student did on CBA problems
    2. Construct a CBA levels summary chart for student.
    3. Develop a set of goals and recommendations for instruction.

CBA Assessment Tasks for Addition and Subtraction

- The problem sheets can be used in individual interviews with children or in class as instructional activities. However, no matter what you choose, it is critical to get the students to write and describe or discuss their strategies. Only then can you use the CBA levels to interpret students' responses and decide on needed instruction.
- The purpose of interviewing students with CBA tasks is to determine how they are reasoning and more specifically, to determine what CBA levels of reasoning they are using for the tasks.

CBA Addition and Subtraction

Student Sheet 1

Problem Set 1

1. Bailey has 4 cubes. Her friend gives her 3 cubes. How many cubes does Bailey have now?

2. Marie has 4 green cubes and 5 white cubes. How many cubes does Marie have altogether?

3. Dennis has 8 cubes. He gives 3 cubes to his friend. How many cubes does he have left?

4. Mary has 5 cubes. Kevin has 7 cubes. How many more cubes does Kevin have?

5. Eric has 6 cubes. Marie has 4 more cubes than Eric. How many cubes does Marie have?

6. Zach has 6 cubes. Brenda gives him 3 more cubes. How many cubes does Zach have?

Instructional Strategies for Addition and Subtraction

- Implementing reasoning strictly verbally is more sophisticated than implementing it concretely or pictorially.
- The strategy is to first help students succeed visually then try to move them to nonvisual strategies, but we can't prematurely rush students to abandon their visual implementations.

<u>Teaching Students at Level 0: Construct Initial Meaning for Addition and Subtraction</u>

- Counting by ones is a prerequisite skill for dealing with beginning addition and subtraction situations
- After students have mastered counting small sets of objects, the next goal is for them to act out simple addition and subtraction problems and solve them by counting all.
- "Understanding" a problem at this stage means being able to act it out and solve it physically.
- It is critical that cubes be available for students to use while working on these early problems.
- Solving problems about unavailable objects by using other objects is more abstract and difficult for students than solving problems that describe the actual objects that are available for manipulation.
- Once a student seems to master problems about cubes, you can move to problems about other things and ask the student to use cubes to represent the problems.
- Once students are able to solve problems by acting them out with physical materials, try to move them to more sophisticated ways of counting all – drawing or using fingers then counting count words.

<u>Teaching Students at Level 1: Operating on Numbers as Collections of Ones.</u>

<u>Level 1.1: Moving to Counting On and Counting Down</u>

- Moving to counting on is a critical milestone in students' use of counting for addition and subtraction.
- Students can correctly produce the sequence of counting words beginning from arbitrary number in a sequence.
- Students recognize that the last number word said in counting m objects is the number m.
- Activities:
    - Dot/Numeral Cards
    - Hidden Squares
    - Dominoes
    - Large/Small Number Combinations

Level 1.2: Moving to Deriving Number Facts

- There are several strategies that you can use to help students derive addition and subtraction facts, including:
  - Ten-Frame Quick Images for Grouping to Ten
  - Using the Commutative Property
  - Deriving Facts from Doubles
  - Relating Addition and Subtraction (Fact Families)

Level 1.3 & 1.4: Moving to Counting by Tens and Ones

- Moving from treating numbers as collections of ones to treating them as collections of tens and ones – so that they can be counted by tens and ones – is a major step for students.
- If a student finds an answer by counting by ones , ask if there is another, faster way to count. If students don't mention counting by tens, ask if counting by ten will help them find how many cubes there are.
- Activities include:
  - Connecting Cubes
  - Paper Strips of Ten Squares
  - Place-Value Blocks
  - Place-Value House Diagram

Teaching Students at Level 2: Moving Toward Sophisticated Counting , then Combining and Separating Place-Value Parts.

Level 2.1: Moving to Counting by Tens in Mid-Decades

- Activities for Addition & Subtraction:
  - Circle Ten-Strip Problems
  - House Problems
- Once students have become proficient using Levels 2.1 and 2.2 reasoning with addition problems, you can have them use this reasoning with subtraction problems.

Level 2.2, 3.1, 3.2: Moving to Combining and Separating Place-Value Parts

- Encourage students to progress to adding and subtracting by decomposing and combining numbers by their place-value parts (ones, tens, hundreds, and so on) using Level 3 reasoning. These place-value parts are combined or separated directly, without counting.
- Multiples-of-ten language (forty) or tens language (4 tens); because the multiples-of-ten language specifically ties units of ten to units of one, it is less abstract than the tens language.
- As they work, observe whether they have to count by tens or can think of the problems in terms of adding tens; 5 tens plus 4 tens is 9 tens or 90, or 50 plus 40 is 90.

Teaching Students at Level 3: Moving Toward Expanded Algorithms

Level 3.3: Moving to Developing Expanded Algorithms

- Before students move to algorithms they should be able to mentally decompose and combine numbers by place value as in Levels 3.3 *without place-value blocks.* (Performance with the blocks is not a reliable indicator that students are ready for algorithms)
- To encourage students to develop the thinking that underlies the algorithms, you might add an extra rule when students are using place-value blocks: whenever there are ten of any one kind of block, you must regroup or trade.
- Students should be encouraged to move to an expanded, conceptually explicit, algorithm as shown below. For many students, these algorithms help them organize their conceptually rich problem solving, increasing students' fluency and lessening their errors.
- The key to helping students learn expanded algorithms meaningfully is to guide students to see these algorithms as organized ways to implement their Level 3 reasoning.

Teaching Students at Level 4: Moving to Traditional Algorithms.

- Before students move to traditional algorithms, they should not only be able to mentally decompose and combine numbers by place value as in Level 3.3 without place-value blocks, but they should have learned expanded algorithms.
- You may need to have a discussion to encourage students to relate the traditional addition and regrouping algorithm to an expanded algorithm.
- Students can also build understanding of traditional algorithms by using place-value blocks *while doing the algorithm*. Doing steps with the blacks at the same time as corresponding steps in the written algorithm encourages students to develop conceptual meaning for the written algorithms.